Mathematics

A conversation with Chris McGrane

Chris McGrane has been teaching for 15 years, across four very different schools. He is Head of Mathematics at Holyrood Secondary, Glasgow - Scotland's largest school. Chris is the author of *Mathematical Tasks: The bridge between teaching and learning*. Chris blogs and shares tasks, including the popular curriculum booklets, on his website: www.startingpointsmaths.com. Chris has also worked as a mathematics consultant for La Salle Education, delivering CPD across the whole of Scotland and Northern England.

In terms of mathematics, at the end of Year 9 you would be looking for three things in a student. Firstly, procedural fluency and a bank of key skills in things like negative numbers, fraction work, basic algebra, percentages, solving equations etc. That is obvious; that is the easy answer. Next comes the conceptual understanding. Once you get to a point with mathematics where you can do it but you no longer understand what is going on, you cannot make the step to the next stage. Some students can get high grades at 16 years old with surface level learning, but they do not actually know very much. As they progress on to the next level, their performance begins to crumble a bit. If you have not really understood what went before, you are going to find it more challenging to build upon it. So you have procedural fluency, you have conceptual understanding and then you have mathematics, which is actually problem solving. Mark McCourt talks about how being a mathematician is a way of being; it is a way of interacting with the universe; it is a way of thinking – and that is what you are looking for, ultimately. Once students leave school, they will forget the mathematics; they will forget some of the understanding. What do you want to leave them with? Well in fact it is going to be some of those soft skills in the long term, and those, perhaps, might transfer into other parts of the curriculum as well. The idea of trying a simpler case, of making a diagram, of coming up with an example of your own – a whole raft of different mathematical behaviours.

You have the basic actions like calculating and following procedures. You have transformative stuff like visualising and representing ideas differently, organising information. One of the key things of mathematics is generalisation: looking for what is the same and what is different and taking the structure out of that, extending ideas beyond what is in front of you. So these are all mathematical things and modelling, predicting, explaining and verifying are all going to be important. I think that often gets lost because it is very easy to spell out your learning intentions, the topics, and how we want kids to learn this, but that meta-mathematics – the stuff that permeates it all – can be lost very easily.

The curriculum must be a route map, but it has to have elements of flexibility built into it; your curriculum cannot just be a list of learning intentions. The curriculum is defined not just by how you teach but by what you ask the students to do with what you have taught them. Essentially, which tasks we give pupils to do matters a lot. It is too easy just to give students screens of procedural questions – 'Here's how you find two-fifths of 40; here's how you find a seventh of 49. Go and do 100 of them' – while the teacher sits back. That is no use. Yes they do need some practice of the basics, but you need to decide the tasks you will use to help them build understanding. What metaphors does a teacher have available to them? Does a teacher know about bar modelling, for instance? Or the option of using an open number line? There is a whole raft of different things which you could be using to teach any particular topic.

There needs to be someone with the overarching vision for the curriculum, who knows their stuff and can pull it all together – even if using a commercially available curriculum. High-quality professional development to give staff the capacity to deliver what you are envisioning is essential. I talked about procedures, the concepts and then the problem solving. Getting students to follow procedures well sounds like it should be an easy matter, but it is hard. So there are pedagogical devices we can use, such as example-problem pairs. If you are not using devices like that when appropriate or if you are not using formative assessment, then you need to upskill the department. But then you want to get into the mathematics specific pedagogical content knowledge. You need to because there is generic stuff out there like Rosenshine, but that is not enough to teach mathematics. We need such approaches as part of our armoury, but we need to be able to go beyond genericism and nail the unique pedagogical approaches that enable students to learn mathematics.

You want this professional development and the curriculum route map to come together. Now, one approach you could take is hinged upon two things: the idea of mastery learning and in-house developed booklets. You will have heard all sorts of stuff about maths mastery; most of it has just bastardised the concept. Mastery learning is not a modern thing. It has a well-established genealogy. Benjamin Bloom in the 1960s did tons of work on mastery learning. With figures like John B. Carroll and his model of school learning – or going further back to people like Carlton Washburn – there is a century of research on 'mastery learning'. It is a rigorous formative assessment cycle.

The problem mastery learning addresses is one that plagues our classrooms and is rooted in the curse of content coverage. It is dead easy to do formative assessment where the teacher will say, 'OK, none of you got that right. That is good to know, but tomorrow we're moving on anyway because the course plan says we've got to.' Mastery is the opposite of that: it is about responsive teaching, ensuring understanding across the class is secure before you move on. That is pivotal. So the mastery idea needs to be embedded, with obvious CPD implications. That involves discussions at department meetings; it is professional reading for colleagues to go away and do and come back and discuss more.

Chris McGrane's booklets

These booklets are key to my approach to teaching mathematics. For every teacher in my department, they spell out a number of things, beginning by spelling out the specific topic. The tasks and the booklets bring the curriculum to life. They embody what we value and what we see as being the essential building blocks of mathematics. Teachers pick up these booklets and they work with them. The booklets are a catalyst for pedagogical change. If we are moving to using an approach which is quite novel, the tasks and examples in the booklets can support the teacher. The booklets explain a unique vision for mathematics. In these booklets, you have got assessment examples as well. Pupils know that if it is in the booklet, it could be in the test.

Consider the order of precedence of operations. Some teachers use ideas like BODMAS, but there are other approaches which come at it more conceptually. If I include a task based upon an area which clearly shows why we multiply before we add, then my teachers see this and it is nudging them to think about a different way of doing it. I always keep saying to them, 'I'm not going to tell you how to teach, but I'm hopefully going to give you other perspectives on it.' However, for some topics, it is about just having a consistent metaphor that you use the whole way through the topic or the termly scheme or the whole year. The idea of algebra tiles as a physical manipulative is based upon the idea of additive inverses, the zero pair: a positive and a negative makes zero. The whole topic of negative numbers – the whole of algebra, even – can be taught from this tiny starting point, and it can be taught consistently. It is the same with the grid model for multiplication, instead of having a traditional column multiplication. People always say, 'Well, that is symptomatic of new mathematics pedagogy, just being fancy for its own sake.' But it is not. It is actually a very forward-facing model which can be built on the whole way through the school right up to the end of A level. It is not optional that you use that pedagogy. It is the way you get consistency. So you are going to find it very hard to use the booklet if you do not teach it that way.

There is procedural content in the booklets, and then there are problem solving tasks and investigative tasks too. All this together is designed to build up and define the curriculum for teachers; and then as pupils progress through this, their skill set develops and improves. Teaching for mastery learning properly sees teacher expectations of pupils go through the roof. You are not moving on unless most of the class have got it; when the teachers realise they can not just opt out – they can not just say, 'Oh, it is the students: they are not good enough' – they fix it. Ensuring that the vast majority of the class have understood what you have taught them is an exceptionally high expectation. You have to believe that the majority of children can actually learn the curriculum and progress in mathematics.

But you have to think about where the students come from, and primary transition is a tough nut to crack. Establishing robust information is a start, so you begin secondary with some diagnosis. And it is not just about establishing what they do not know; often you can waste a bit of time in early secondary telling them stuff they already know. Maths teachers are guilty of that. Basic angle work is a classic one, as is symmetry – why the hell are you doing the line symmetry, paper-folding butterflies with 12-year-olds? You need to cut some content because they know it or they know enough about it. You could spend three weeks playing with factors, primes and multiples because there is so much you can do there, but one week will be enough because it is not going to make or break their mathematical futures. Whereas the lack of fluency on negative numbers or fractions will make or break it later on, so you need to put the time into that.

Exemplifying mastery learning: negative number booklet

This booklet is used very early on in the first year of secondary. We know that they do know stuff about negative numbers already. We start with this page here, but they probably should be able to do this already.

Getting Familiar with Negatives

1. Write the following sets of numbers in ascending order:
   - (a) 3, -5, 1, 0, -2, 4
   - (b) -1, 8, -5, 2, -9, -4, 3
   - (c) -1, -7, -2, 5, -6, 1
   - (d) 10, -7, -3, 5, -9, -2 ,-12
   - (e) 21, -3, 16 ,-19, -15, 23, -30
   - (f) -25, 35, 15, -5, 25, -45, 20
   - (g) 129, 101, -11, -111, 92, -91, 133, -29

2. Place the correct sign, < or >, between each pair of numbers
   - (a) 3 □ 1
   - (b) 2 □ 7
   - (c) 10 □ 11
   - (d) 8 □ 5
   - (e) 33 □ 25
   - (f) 28 □ 21
   - (g) 102 □ 99
   - (h) 110 □ 113

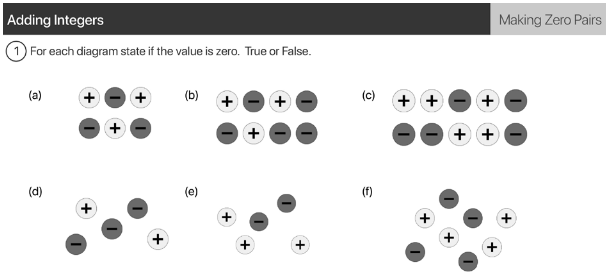
3. Place the correct sign, < or >, between each pair of numbers
   - (a) -3 □ 2
   - (b) 4 □ -1
   - (c) -3 □ -1
   - (d) -19 □ 15
   - (e) -20 □ -30
   - (f) -8 □ -11
   - (g) -12 □ -9

4. The temperature in Oban at 8 p.m. was 4 °C. At midnight the temperature had fallen by 7°C. What was the temperature at midnight?

5. The average temperature in Moscow in January is -10 °C.
   The average temperature in Sydney is 23 °C. What is the difference in temperature?

You see, if significant numbers of the class are not comfortable with that first page, then you are probably going to have to stop and teach some of that; but at the outset, I do not want to be teaching any of that to begin with – I just want to see if they can do it. So I am going to diagnose where they are to begin with. Certainly we have found that 80% of the year group can do that stuff off the cuff and do not require additional input. I think it is important to check it, but we do not need to waste time teaching it, if that makes sense, because the assumption that they will know is fairly accurate.

Next up is the idea of making zero pairs, and this is the metaphor we use which spans the whole six years of secondary school. It is the basics of conceptual understanding.

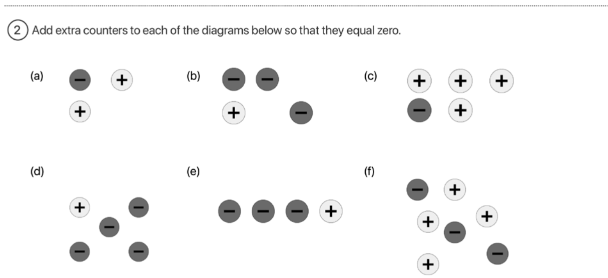


For each of these diagrams, is the value zero? In task 1 (a), it is, because for every positive there is a corresponding negative, so that is worth zero. In diagram 1 (b), the value is not zero as there are more negatives than positives.

The booklet also directs the pedagogy of the teacher. It is not just about having a good task; it is knowing how to use the task. So having departmental meetings and discussions about, 'What was the purpose of this task? What were the design features of it? Why was it put in here?'

Going back to question 1 (b) above, if it is not zero, ask the student, 'Well, what is it then?' And then talk them through the columns. 'Well, that first column is a positive and negative which makes zero. The next two columns are zeros. But what have I got left at the end? I've got two negatives, so what number is that? It is negative two.' Students will never have to answer a question like that in mathematics, but it is a micro step on the way to proper understanding.

Next task, add some counters in yourself so that the value of each of these is zero.



So again in 2 a), to make it zero, we need to add in a negative.

Then we are going into the symbolic, so I have got five plus negative five – what does that make? It makes zero. Then we ask, 'Three plus *what* makes zero?'

Now the kids at this stage in Scotland have not seen algebra, but question 3 (f) below, '*a* plus *what* would give me zero? Well you are generalising; you are a mathematician. Is it negative *a*?' And they think, 'Wow, I got that! I'm a mathematician!' and they feel quite good about themselves.

3 Complete each statement below:

(a) $5 + {}^{-}5 = \square$

(b) $3 + \square = 0$

(c) ${}^{-}7 + \square = 0$

(d) $\square + {}^{-}4 = 0$

(e) $\square + 2 = 0$

(f) $a + \square = 0$

And what we are getting at here is the idea of additive inverse – the idea that for every number, there is an additive inverse of it which you can add to it to get zero. And that accurate language is important.

(4) Write down the additive inverse of each number

(a)	5	(b)	-4
(c)	-40	(d)	12
(e)	$\frac{1}{2}$	(f)	-0.56
(g)	*a*	(h)	-*a*

So what have we added to five to get zero? We have negative five. What are we adding to negative four to get zero? We are adding positive four, and so on. So the teacher might use that as a starting point and go off and do some extra stuff on the board if the students are not getting it. But you can see here how the tasks are designed, the micro-steps we take. We are making no assumptions and we are building steadily, and then you get this:

Adding Integers — Representing Addition

(1) Complete the table below.

Question	Re-written	Using Tiles	In words	Answer
5 + (-2)	+5 + -2	+ + + + + − −	positive 5 plus negative 2	3
4+ (-3)				
(-5) + 3				
-2 + 1				
3+ (-4)				
2+ -5				
(-3)+ 4				
-2+ 3				
(-2) + (-3)				
-3 + -4				

This is the addition of negative integers. There are positive numbers in there too. And that example at the top is not for the students because they are not teaching themselves. This example at the top is a feature; it is a reminder to the teacher of how to do this. So what we have here in the first example question is five plus negative two. This idea of signed numbers with that little kind of superscript positive and negative in the front was quite a common thing back in the 1970s and 1980s. It lends itself quite well to this, because if you have got five of those positive ones and two of those negative ones, you are removing the ambiguity between, 'Is that the sign of the number and is it an addition or a subtraction?' By adding a little layer more, you are actually removing a concerning ambiguity. Later on, students can jump to this without having to add the extra +/– signs, but initially you are making that explicit.

So, you have got the diagram, you have got it in words and there is the answer. That is it; now just practise. Sometimes the brackets are there, and sometimes they are not. In the past, you would not have all this cognitive support here, all the headings and the structured building blocks. You would have had the list of questions and have been told to go and do them.

When I was sold on this approach, I had quite a low prior attainment class in my last school. I taught them negative numbers using this, and they were as good as a top set on negative numbers. I had taught it other ways for years – out in the corridor, running up and down on cards on the floor, and all sorts of stuff. But this was the best way I had encountered to date.

I discovered this by looking at some textbooks from Singapore a few years ago, and then I heard different people talk about it in different ways, and there are debates out there about how you do it. But you have to take all that is out there and then you try it on your class; that way, you get ownership of it. Because it has to work in your context.

By the time they get to this page, they have to be super resilient. I love this task.

Subtracting Integers Generalising

(1) Create an example of an addition which belongs in each box.
Some boxes are not possible to fill in - decide which ones.

	Difference is positive	Difference is **zero**	Difference is **negative**
positive - positive			
negative - positive			
positive - negative			
negative - negative			

Students have got to generate their own instances; they've got to begin to get a feel for negative numbers: Give me a positive and a positive whose sum is positive. Well, two plus four – that works. Give me a positive and a positive whose sum is zero. Oh hold on a minute, I cannot do that. Can you give me two positive numbers whose sum is negative? No, I cannot do that. What about a negative plus a positive which has a positive sum? Well negative two add five does give me a positive: it is three. What about a negative? I have a positive which gives me zero; we can come up with that. And what about a negative and a positive which gives me a negative?

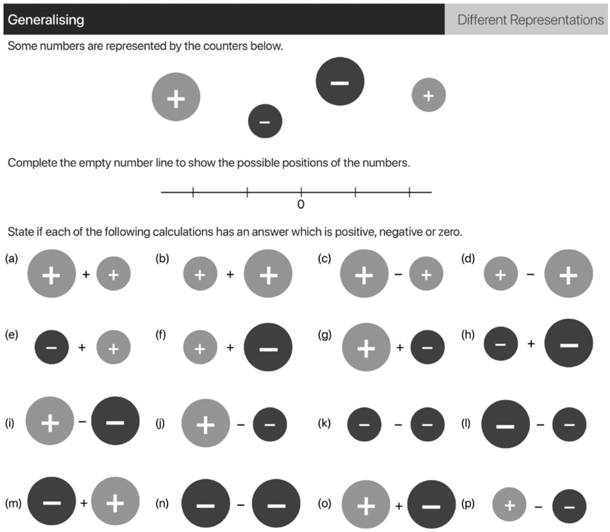
So they begin to think in a principles-based way. It is about taking them from doing my mathematics to making their own instances of it and beginning to get a sense of the structure which is underlying all this. You get some good conversations, because not all the boxes are possible. You do find that they sometimes break their own understanding of it, which is good.

Sometimes the tiles are the best visual model; other times, the number line is better. So what we are trying to do here is to give students two ways of thinking.

You are always trying to clarify things for students. Take the idea of subtraction. Subtraction is a controversial one because you will notice that every time I have said 'negative five' or 'negative two', not the colloquial, 'It was minus four outside last night.' Minuses have a very precise meaning. Consider 'five minus two'. It is, more precisely, 'positive five minus positive two'. So minus is the action of subtraction; it is not a minus number. That symbol means we add the additive inverse of that number. We make things a little bit harder in the short term for the long-term gain. People want to smooth the path all the time in maths – you want to make it easy for them. That is not necessarily the right thing if we are going to persevere in early secondary. If you can get a good robust forward-facing method that will always work, which is a little bit harder, it is worth sacrificing a couple of days now to never see that mistake at GCSE.

Tony Gardner wrote a beautiful book called *Teaching Mathematics at Secondary*. When he talks about a forward-facing method, he means a metaphor which does not break, a metaphor which is axiomatically robust. It is actual mathematics. It is not a trick and it will always work. The classic example would be solving equations. People would trot out, 'Change side; change sign', and it is a pile of nonsense because it works for a very limited range of equations. When you start getting into stuff like exponentials and derivatives further up the school, it is a metaphor which crumbles. Students have to see algebra completely anew, which is the last thing you want to be doing when you are trying to marshal much more difficult ideas. So you want a method, a metaphor, a model which works the whole way through. We will have talked about this for quite a bit in a departmental meeting. We will have had informal conversations in the department base. It is not just a case of, 'Here is this pedagogical approach; now go and run with it.' You are effectively an NQT with this topic when you go back and do it from scratch; it is the first time you have taught such a fundamental topic this way. So it can get messy, but it tends to work.

Some students are going to need more time just to master some of the basic material, but others will race through. So having extra stuff there gives the higher attainers something which is meaningful. For instance, in the negative numbers booklet, this task is quite powerful:



We are building on our counters, but now we are giving them a size: big negative, small negative, big positive, small positive. And the students go through this and they have got to decide whether it is positive, negative or zero. Again, what I am trying to do is get at the structure – what is actually going on. So for instance, if I have a small positive but add a big negative, I am talking in general; I am not using specific instances of numbers. What are we getting out of this? The students have got to come up with a case to help them: 'Well let us say that is two but that is negative five. Oh, I can see that the answer must be negative.' All this does is get them to think harder, and it is this idea of building up a schema, the idea of understanding, making more connections. My philosophy is that I am never going to teach you negative numbers again; we are not going to do it again in the year after. In my NQT year, I had a class who had been taught negative numbers six years in a row, and on the first day I went to teach them negative numbers they said, 'We've done this every single year, Sir.' I gave them five or six questions and they still could not do it. So I thought, 'If we teach it right, we should not need to be repeating this every single year because it is not even something you need to plan for them to retain: it comes up all the time in other topics. It is not like some obscure branch of maths which we have not done for a long while but which we will need for the test. It is in everything; it permeates so much.

The principles of mastery learning are challenging. You are building up the basic concept and you are coming at it from myriad directions. Then at the end, there has got to be enrichment available, because it is part of the mastery model. So you have a little formative assessment at the end of the topic and the topic is the booklet; some pupils will pass, so they do some extra enrichment. Those who have not passed will do some remediation.

The extra enrichment is gorgeous. Let us have a look at this:

Enrichment	Subtraction methods

Normally we do subtraction as follows:

436 – 258

$$\begin{array}{r} {}^{3}\,{}^{12}\;\;\; \\ \cancel{4}\,\cancel{3}\,{}^{1}6 \\ -\;2\,5\,8 \\ \hline 1\,7\,8 \\ \hline \end{array}$$

What does this mean?

1 **hundred** 7 **tens** 8 **ones**

We can write this as:

100 + 70 + 8 **= 178**

There is much nicer way of doing subtraction, now that we understand negative numbers!

436 – 258

$$\begin{array}{r} 4\;\;3\;\;6 \\ -\;\;2\;\;5\;\;8 \\ \hline 2\;-2\;-2 \\ \hline \end{array}$$

200 – 20 – 2 = 178

You can see that 4 – 2 = 2 and 3 – 5 = -2 and that 6 – 8 = -2.

What does this mean?

2 **hundreds** -2 **tens** -2 **ones**

We can write this as:

200 - 20 - 2 **= 178**

So 436 minus 258. Normally we work right to left, but that is idiosyncratic because we should be going left to right, because it is how we read. So why do we not just go left to right: four minus two is two. Three minus five is negative two. Six minus eight is negative two. So what have I got here? I have two 100s. It is going back to place value. I have negative two 10s and I have negative two 1s. So, it is 200 minus 20 minus 2 which is 178. Boom! Those students who have done well, you are trying to enrich them, you are trying to imbue them with that bit of passion of being a mathematician. You have to draw your students into the power and the story and the elegance of mathematics. There is a great quotation from Einstein that says that 'mathematics is the poetry of logical ideas'.

I am training my colleagues as they are working through the booklets; their own understanding of the pedagogy of the mathematical principles is being deepened.

If we want to convey the beauty, the magnificence, the internal elegance and structures behind any subject, and in particular maths, children have to catch it from us. And when we are using this kind of language that comes from a deep space, it becomes utterly irresistible.

Background: mathematics

From 3000 BCE, arithmetic, algebra and geometry were developed in Mesopotamia for commerce, to record time and to work out calendars. The Pythagoreans were the first to refer to mathematics and were mostly concerned with deductive reasoning. Chinese mathematics developed a place value system and the concept of negative numbers. The Indo-Arabic numeral system was invented between the first and fourth centuries in India and adopted by Arabic mathematics in the ninth century. Algebra was invented by the influential Persian polymath Al-Khwarizmi.

It is worth quoting the purpose of the subject from the national curriculum programme of study:

> Mathematics is a creative and highly interconnected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems. It is essential to everyday life, critical to science, technology and engineering, and necessary for financial literacy and most forms of employment. A high-quality mathematics education

> therefore provides a foundation for understanding the world, the ability to reason mathematically, an appreciation of the beauty and power of mathematics, and a sense of enjoyment and curiosity about the subject.[1]

The national curriculum for mathematics aims to ensure that

> all pupils become fluent in the fundamentals of mathematics, including through varied and frequent practice with increasingly complex problems over time, so that pupils develop conceptual understanding and the ability to recall and apply knowledge rapidly and accurately; reason mathematically by following a line of enquiry, conjecturing relationships and generalisations, and developing an argument, justification or proof using mathematical language; can solve problems by applying their mathematics to a variety of routine and non-routine problems with increasing sophistication, including breaking down problems into a series of simpler steps and persevering in seeking solutions.
>
> Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas. The programmes of study are, by necessity, organised into apparently distinct domains, but pupils should make rich connections across mathematical ideas to develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems. They should also apply their mathematical knowledge to science and other subjects.

Once the importance statements have been revisited, it is helpful for subject leaders and coordinators to discuss and agree with colleagues the reason why their subject, in this case maths, is important for the pupils in their school. One way of doing this is to draw on a quote, in this case from Albert Einstein: ‘Pure mathematics is, in its way, the poetry

1 www.bit.ly/3mrWjoT

of logical ideas.' This kind of prompt allows us to formulate our way of stating the importance of the subject. We might agree or disagree with such a statement and in doing so come to a form of words which expresses our view of the importance of this subject, in this school. This moves us away from the territory of 'We teach this subject because of the SATs or GCSEs'. While the external tests and exams are important, they are not the totality of the subject.

Professional communities

Subject associations are important because at the heart of their work is curriculum thinking, development and resources. There are a number of subject associations for mathematics and the National Centre for Excellence in the Teaching of Mathematics[2] lists these on its website. Any member of staff with responsibility for a subject should be a member of the relevant subject association, and this should be paid for by the school.

Twitter subject communities are important for the development of subject knowledge, because it is here that there are lively debates about what to teach, how to teach and the kinds of resources that are helpful. For maths, it is worth following the NCETM on Twitter[3] and the hashtag #mathscpdchat.

LINKS

Mathematical Association – www.m-a.org.uk
National Centre for the Teaching of Excellence in Mathematics – www.ncetm.org.uk
La Salle Education – www.completemaths.com

2 www.bit.ly/2W5tHpY
3 www.twitter.com/ncetm

An overview of the Holyrood School key stage 3 mathematics curriculum

	Year 7 Mixed Attainment Block Used with pupils on initial arrival at Secondary school. This is a mixed attainment booklet which runs for a few weeks. We did the quizzes at the back on an ongoing basis to help determine the appropriate working level for pupils.
Phase 1	• Area and Length (basics) • Order of Operations • Basic Fraction Work • Diagnostic Quizzes on basic number
Phase 2	• Arithmetic Essentials (1) • Integers (1) • Factors & Primes • Angles (1) • Fractions (1)
Phase 3	• Arithmetic Essentials (2) • Integers (2) • Fractions (2) • Algebra (1) • Statistics
Phase 4	• Angles (2) • Algebra (2) • Percentages • Solving Equations (1)
Phase 5	• Ratio • Solving Equations (2) • Area (2) and Volume • Coordinates and Sequences

Three documents for your senior leader line manager to read about mathematics

1. McCourt, M. (2019) *Teaching for mastery*. Woodbridge: John Catt Educational.
2. Cockcroft, W. H. (1982) <I>Mathematics counts</i>. London: Her Majesty's Stationery Office.
3. Askew, M., Rhodes, V., Brown, M., Wiliam, D. and David Johnson, D. (1997) *Effective teachers of numeracy: report of a study carried out for the Teacher Training Agency*. London: King's College, University of London.

Five questions for your senior leader line manager to ask you about mathematics

1. How does whole-school learning and teaching policy correlate with what you know about mathematics teaching? Is it complementary or does it impede it?
2. How and why have you structured your classes the way you have? e.g. setting, mixed attainment, etc.
3. What can I do to support improvement in the mathematics department?
4. What can I do to support you in developing leadership and management skills?
5. Would you like me to attend/observe department meetings?